Journal of Applied Mechanics and Technical Physics, Vol. 39, No. 6, 1998

ANALYSIS OF STABILITY OF A THIN LAYER OF GRANULAR MATERIAL MOVING ON AN INCLINED PLANE

Yu. A. Berezin and L. A. Spodareva

UDC 532.536

The stability of a layer of a granular medium on an inclined plane has been studied within the framework of the model of a non-Newtonian fluid with an index of 2, which ensures the experimentally found quadratic dependence of the shear stress on the shear rate. It is shown analytically and numerically that these flows are stable or unstable depending on the value of the generalized Reynolds number relative to the critical value equal to $5\cot\alpha$.

Interest in the laws of motion of granular media stems from their abundance in nature and various technological processes [1, 2]. It is well known that the shear stress in rapid granular shear flows is proportional to the shear rate squared, and granular materials are often considered as a non-Newtonian medium.

In [3], we studied a two-dimensional flow of a layer of such a material with a free surface, which moves slowly on a rough inclined plane, based on the model of an incompressible non-Newtonian fluid with an index 2. The assumption of slow motion corresponds to the neglect of inertial components in the equations of motion. The equation obtained for the free surface of the layer in case of small but finite amplitudes reduces to the Burgers equation whose solutions are stable to small perturbations within the entire range of wavelengths (or wavenumbers). For rapid flows, it is necessary to take into account the components of the equations of motion that correspond to flow acceleration, which can alter the character of stability. Exactly these issues are discussed in the present paper.

As in [3], we use the governing equations

$$\begin{gathered}\rho(u_t + uu_x + vu_y) = -p_x + \rho g \sin\alpha + (\sigma_{xx})_x + (\tau_{xy})_y,\\ \rho(v_t + uv_x + vv_y) = -p_y - \rho g \cos\alpha + (\tau_{yx})_x + (\sigma_{yy})_y, \quad u_x + v_y = 0.\end{gathered} \tag{1}$$

The x axis is directed along the inclined plane and the y axis is directed across the plane. We assume that the longitudinal scale L_0 is significantly greater than the transverse scale H_0 ($\varepsilon = H_0/L_0 \ll 1$). Then the transverse velocity v is significantly smaller than the longitudinal velocity u, but $v_y \sim u_x$ (as follows from the continuity equation), and the pressure can be considered as hydrostatic. Since the flow is mainly longitudinal, we take into account only the shear stress component $\tau_{xy} = \tau_{yx}$ which is equal to $\mu|u_y|u_y$, where μ is the dynamic viscosity or a measure of the consistency of the medium. As a result, Eqs. (1) take the form

$$u_t + uu_x + vu_y = -p_x/\rho + g\sin\alpha + \nu(|u|_y u_y)_y, \quad p_y = -\rho g\cos\alpha, \quad u_x + v_y = 0. \tag{2}$$

Here $\nu = \mu/\rho$ and $H = H(x,t)$ is a function that describes the shape of the free surface.

We supplement these equations by the boundary conditions $u = v = 0$ on the inclined plane for $y = 0$ and $p = 0$, $\tau_{xy} = 0$, and $H_t + uH_x = v$ on the free surface of the layer for $y = H(x,t)$. Since the shear stress is τ_{xy}, the equality of this shear stress to zero on the free surface is equivalent to the equality $u_y = 0$ for $y = H(x,t)$.

Institute of Theoretical and Applied Mechanics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 39, No. 6, pp..113–117, November–December, 1998. Original article submitted April 3, 1997.

To analyze system (2), we use a common procedure of transition to equations integrated over the thickness of the layer under consideration, which is described in detail, for instance, by Nakoryakov et al. [4] for a viscous Newtonian fluid. Appropriate transformations using the above boundary conditions lead to

$$H_t + (H\langle u\rangle)_x = 0,$$
$$(H\langle u\rangle)_t + (H\langle u^2\rangle)_x = gH(\sin\alpha - H_x\cos\alpha) - \nu u_y^2(0), \tag{3}$$

where

$$\langle u\rangle = H^{-1}\int_0^H u\,dy, \qquad \langle u^2\rangle = H^{-1}\int_0^H u^2\,dy.$$

Now we have to find a relationship between the mean longitudinal velocity and its mean square. We use a steady-state-type profile as an approximation, i.e.,

$$u(x,y,t) = u_s[1-(1-y/H)^{3/2}], \qquad u_s = (2/3)(gH^3\sin\alpha/\nu)^{1/2},$$

and find $\langle u\rangle = 3u_s/5$ and $\langle u^2\rangle = 5\langle u\rangle^2/4$. Substituting these relations into (3), we obtain the sought equations

$$H_t + Q_x = 0, \quad Q_t + 5(\langle u\rangle Q)_x/4 = -(g/2)(H^2)_x\cos\alpha + gH\sin\alpha - 25\nu Q^2/(4H^4). \tag{4}$$

Here $Q = \int_0^H u dy = \langle u\rangle H$ is the volume discharge.

On the basis of Eqs. (4), we analyze the stability of a uniform flow $H = H_0$, $Q = Q_0 = (2/5)(gH_0^5\sin\alpha/\nu)^{1/2}$, and $u_0 = Q_0/H_0$ to infinitesimal periodic perturbations. For this we write $H = H_0 + h$, $Q = Q_0 + q$ ($h \ll H_0$, $q \ll Q_0$), represent the perturbations as $q, h \sim \exp i(kx - \omega t)$, where $\omega = \omega_r + i\gamma$ is the complex frequency, and use the condition of solvability of the system of algebraic equations for perturbation amplitudes. As a result, we obtain a dispersion equation that relates the frequency and the wavenumber k. In dimensionless variables this equation has the form

$$\omega^2 - (5/2)(k - 5i/\varepsilon\,\mathrm{Re})\omega + (5/4)(1 - 5\cot\alpha/\mathrm{Re})k^2 - 125\,ik/(4\varepsilon\,\mathrm{Re}) = 0. \tag{5}$$

The scales of frequency and wavenumber are u_0/L_0 and $1/L_0$, respectively, and $\mathrm{Re} = H_0^2/\nu$ is an analog of the Reynolds number for the medium considered. In a long-wave approximation ($k \ll 1$), the solutions of Eq. (5) can be found analytically. Indeed, substituting the frequency in the form of an expansion with respect to the wavenumber powers $\omega = \omega_0 + k\omega_1 + k^2\omega_2$ and equating separately the terms with equal powers of k to zero, we obtain

$$\omega^{(1)} = 5k/2 + 0.1i\varepsilon(\mathrm{Re} - 5\cot\alpha)k^2, \quad \omega^{(2)} = -i[12.5/(\varepsilon\,\mathrm{Re}) + 0.1\varepsilon(\mathrm{Re} - 5\cot\alpha)k^2].$$

The complex frequency of the first harmonic has a nonzero real part proportional to the wavenumber and an imaginary part proportional to the wavenumber squared. The sign of the imaginary part depends on the Reynolds number: the growth rate of the first harmonic is $\gamma^{(1)} \sim (\mathrm{Re} - \mathrm{Re}_*)k^2 > 0$ for $\mathrm{Re} > \mathrm{Re}_* = 5\cot\alpha$ and $\gamma^{(1)} < 0$ for $\mathrm{Re} < \mathrm{Re}_*$.

The complex frequency of the second harmonic has a zero real part and a negative imaginary part. Thus, in the long-wave range the first harmonic propagates with the phase velocity $\omega^{(1)}/k = 5/2$ and is unstable (stable) for $\mathrm{Re} > \mathrm{Re}_*$ ($\mathrm{Re} < \mathrm{Re}_*$), and the second harmonic does not propagate and decays (is stable).

For arbitrary wavenumbers of perturbations, having separated the real and imaginary parts of the complex frequency, we obtain from Eq. (5)

$$\omega_r = \frac{2.5(\gamma + a)k}{2\gamma + a}, \qquad a = \frac{12.5}{\varepsilon\,\mathrm{Re}}, \qquad k^2 = \frac{0.8\gamma(\gamma + a)(2\gamma + a)^2}{(1 - 5\,\mathrm{Re}^{-1}\cot\alpha)(2\gamma + a)^2 - 5\gamma(\gamma + a)}.$$

In the limit, as $k \to 0$, the growth rates are $\gamma^{(1)} \to 0$ and $\gamma^{(2)} \to -12.5/(\varepsilon\,\mathrm{Re})$. For very short waves the limiting values of the growth rate $\gamma 1$ and the decrement $\gamma 2$ can be determined by equating to zero the denominator of the expression $k^2(\gamma)$. For example, for $\varepsilon = 0.1$ and $\alpha = 45°$ we have

$$\gamma 1 = (62.5/\mathrm{Re})(\sqrt{5\,\mathrm{Re}/(\mathrm{Re}+20)} - 1), \qquad \gamma 2 = -(62.5/\mathrm{Re})(\sqrt{5\,\mathrm{Re}/(\mathrm{Re}+20)} + 1).$$

For small deviations of the Reynolds number from the critical value, the growth rate of the first harmonic is $(\mathrm{Re} - \mathrm{Re}_*)k^2$. It increases monotonically from zero to $\gamma 1$ with increasing wavenumber, and the growth rate of the second harmonic varies monotonically from $\gamma^{(2)} = -12.5/(\varepsilon\,\mathrm{Re})$ to $\gamma 2$.

We consider a weakly nonlinear case. This means that substituting $H = H_0 + h$ and $Q = Q_0 + q$ into Eqs. (4) we retain, along with linear terms, the quadratic ones with respect to perturbations q and h. As a result, we obtain

$$h_t + q_x = 0,$$

$$q_t + \Big(\frac{5Q_0}{2H_0}\Big)q_x - \Big(\frac{5Q_0^2}{4H_0^2}\Big)h_x - g(h\sin\alpha - H_0h_x\cos\alpha) + \frac{25Q_0^2}{2H_0^2\,\mathrm{Re}}\Big(\frac{q}{Q_0} - \frac{2h}{H_0}\Big)$$
$$= -\frac{5Q_0}{2H_0}\Big(\frac{q}{Q_0} - \frac{h}{H_0}\Big)q_x + \frac{5Q_0^2}{2H_0^2}\Big(\frac{q}{Q_0} - \frac{h}{H_0}\Big)h_x - ghh_x\cos\alpha - \frac{25Q_0^2}{4H_0^2\,\mathrm{Re}}\Big(\frac{q^2}{Q_0^2} - \frac{8qh}{Q_0H_0} + \frac{10h^2}{H_0^2}\Big).$$

We differentiate the first equation with respect to time t and the second equation with respect to the x coordinate and obtain the equation

$$h_t + c_0h_x + \frac{2H_0^2\,\mathrm{Re}}{25Q_0}[h_{tt} + c_0h_{xt} + (0.2c_0^2 - gH_0\cos\alpha)h_{xx}]$$
$$= \frac{2H_0^2\,\mathrm{Re}}{25Q_0}\Big[\frac{5}{2H_0}(qq_x)_x - \frac{5Q_0}{2H_0^2}(qh)_{xx} + \Big(\frac{5Q_0^2}{2H_0^3} + g\cos\alpha\Big)(hh_x)_x + \frac{25}{2H_0^2\,\mathrm{Re}}\Big(qq_x - \frac{4Q_0}{H_0}(qh)_x + \frac{10Q_0^2}{H_0^2}hh_x\Big)\Big],$$

where $c_0 = 5Q_0/2H_0 = (5/2)u_0$.

We suppose the nonlinearity to be small; therefore, we can assume that $q = c_0h$ in nonlinear terms, and then we have

$$h_t + c_0h_x + \Big(\frac{3c_0}{2H_0}\Big)hh_x + \Big(\frac{H_0\,\mathrm{Re}}{5c_0}\Big)\Big(\frac{\partial}{\partial t} + \frac{c_1\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial t} + \frac{c_2\partial}{\partial x}\Big)h = \Big(\frac{c_0\mathrm{Re}}{5}\Big)\Big(\frac{9}{10} + \frac{\cot\alpha}{\mathrm{Re}}\Big)(hh_x)_x.$$

Here $c_{1,2} = (c_0/2)(1 \pm \sqrt{1/5 + 4\cot\alpha/\mathrm{Re}})$. Substituting $\partial/\partial t \to -c_0\partial/\partial x$ in terms with mixed derivatives, which does not change the order of accuracy, we obtain the equation

$$h_t + c_0h_x + \Big(\frac{3c_0}{2H_0}\Big)hh_x + \frac{c_0H_0\,\mathrm{Re}}{25}\Big(1 - 5\cot\frac{\alpha}{\mathrm{Re}}\Big)h_{xx} = \frac{c_0\mathrm{Re}}{5}\Big(\frac{9}{10} + \cot\frac{\alpha}{\mathrm{Re}}\Big)(hh_x)_x.$$

The last term in the left-hand side corresponds to positive or "negative" viscosity depending on whether the Reynolds number is smaller or greater than $\mathrm{Re}_* = 5\cot\alpha$. As in the linear case, the flow is unstable for high values $\mathrm{Re} > \mathrm{Re}_*$ and stable for small values $\mathrm{Re} < \mathrm{Re}_*$. To study the nonlinear evolution of spatially localized initial perturbations, we numerically solved Eqs. (3), which have the dimensionless form

$$H_t + Q_x = 0, \quad Q_t + (5/4)(\langle u\rangle Q)_x = -(25\cot\alpha/4\,\mathrm{Re})HH_x + (25/4\varepsilon\,\mathrm{Re})(H - Q^2/H^4).$$

The solution was obtained using an explicit conditionally stable scheme and taking into account the sign of the averaged longitudinal velocity. The mass and momentum fluxes are approximated by one-sided differences, and the pressure gradient is approximated by central differences. For simplicity, the initial perturbation was a triangle of height $H_1 = 0.1$ and base width $\Delta = 2$ in dimensionless units. At the initial moment of time this triangle, located on a uniform layer with an undisturbed surface $H = 1$, suddenly becomes free, and the perturbation, changing in shape, starts to move along the layer. The calculations were conducted for $\varepsilon = 0.1$ and $\alpha = 45°$.

Figure 1 shows profiles of elevation of the free surface of a granular layer as functions of the coordinate along the inclined plane at different times and for different Reynolds numbers:

(a) for the Reynolds number smaller than the critical value $\mathrm{Re} = 4$ (according to the linear analysis, the critical value is $\mathrm{Re}_* = 5$ for $\alpha = 45°$), the initial perturbation propagates downstream with a noticeable

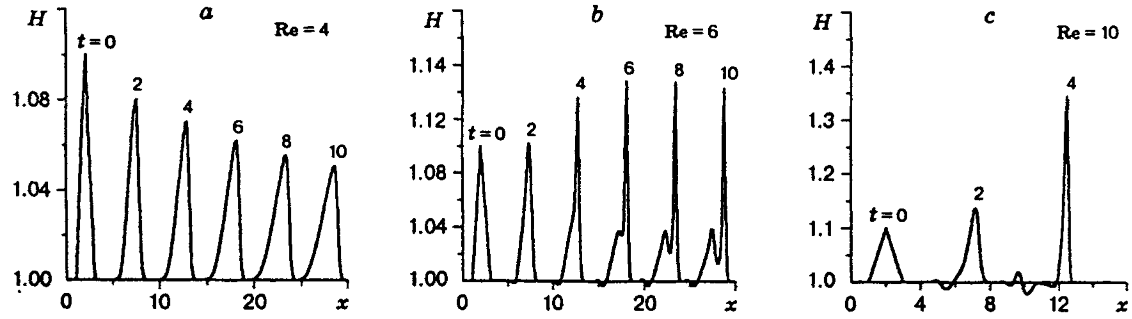

Fig. 1

decrease in the amplitude and an increase in the width;

(b) Re = 6, the initial elevation loses its symmetric shape in time, and a saw-tooth profile arises;

(c) the change in the initial profile begins in earlier stages of evolution (Re = 10).

Thus, for a layer of a dry granular material considered as a non-Newtonian fluid with an index of 2 that moves on a rough inclined plane, we derived equations for the free-surface shape and the longitudinal momentum which are averaged over the depth of this layer. Based on the linear analysis, it is shown that infinitesimal perturbations are stable (unstable) for $\mathrm{Re} < \mathrm{Re}_*$ ($\mathrm{Re} > \mathrm{Re}_*$). For the case of weak nonlinearity the problem reduces to one equation for the free surface. Using a numerical solution, the evolution of initial perturbations of finite amplitude is considered for subcritical and supercritical Reynolds numbers.

REFERENCES

1. S. B. Savage, "Gravity flow of cohesionless granular materials in chutes and channels," *J. Fluid Mech.*, **92**, 53–96 (1979).
2. K. Hutter and K. R. Rajagopal, "On flows of granular materials," *Cont. Mech. Thermodyn.*, **4**, No. 2, 82–141 (1994).
3. Yu. A. Berezin and L. A. Spodareva, "Slow motion of a granular layer on an inclined plane," *Prikl. Mekh. Tekh. Fiz.*, **39**, No. 2, 117–120 (1998).
4. V. E. Nakoryakov, B. G. Pokusaev, and I. R. Shreiber, *Propagation of Waves in Gas- and Vapor-Liquid Media* [in Russian], Inst. of Thermal Physics, Novosibirsk (1983).